

Branching Random Walk in an inhomogeneous breeding potential

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Abstract

We consider a continuous-time branching random walk in the inhomogeneous breeding potential $\beta|\cdot|^p$, where $\beta > 0$, $p \geq 0$. We prove that the population almost surely explodes in finite time if $p > 1$ and doesn't explode if $p \leq 1$. In the non-explosive cases, we determine the asymptotic behaviour of the rightmost particle.

1 Introduction and main results

We consider a branching system with single particles moving independently according to a continuous-time random walk on \mathbb{Z} . The random walk makes jumps of size 1 up or down at constant rate $\lambda > 0$ in each direction. A particle currently at position $y \in \mathbb{Z}$ is independently replaced by two new particles at the parent's position at instantaneous rate $\beta|y|^p$, where $\beta > 0$ and $p \geq 0$ are some given constants.

We denote the set of particles present in the system at time t by N_t . If $u \in N_t$ then the position of a particle u at time t is X_t^u and its path up to time t is $(X_s^u)_{0 \leq s \leq t}$. The law of the branching process started with a single initial particle at $x \in \mathbb{Z}$ is denoted by P^x with the corresponding expectation E^x and the natural filtration of the process is denoted by $(\mathcal{F}_t)_{t \geq 0}$.

Let us define the explosion time of the population as

$$T_{\text{explo}} = \sup\{t : |N_t| < \infty\}.$$

We have the following dichotomy for T_{explo} in terms of p , the exponent of the breeding potential.

Theorem 1.1 (Explosion criterion). *For the inhomogeneous BRW started at any $x \in \mathbb{Z}$:*

- a) *If $p \leq 1$ then $T_{\text{explo}} = \infty$ P^x -a.s.*
- b) *If $p > 1$ then $T_{\text{explo}} < \infty$ P^x -a.s.*

Let us also define the process of the rightmost particle as

$$R_t := \sup_{u \in N_t} X_t^u, \quad t \geq 0.$$

For $p \in [0, 1]$, we prove the following result about the asymptotic behaviour of R_t .

Theorem 1.2 (Rightmost particle asymptotics). *For the inhomogeneous BRW and any $x \in \mathbb{Z}$:*

- a) *If $p = 0$ then*

$$\lim_{t \rightarrow \infty} \frac{R_t}{t} = \lambda(\hat{\theta} - \frac{1}{\hat{\theta}}) \quad P^x\text{-a.s.}, \quad (1.1)$$

where $\hat{\theta}$ is the unique solution of

$$\left(\theta - \frac{1}{\theta}\right) \log \theta - \left(\theta + \frac{1}{\theta}\right) + 2 = \frac{\beta}{\lambda} \quad \text{on } (1, \infty) \quad (1.2)$$

b) If $p \in (0, 1)$ then

$$\lim_{t \rightarrow \infty} \left(\frac{\log t}{t}\right)^{\hat{b}} R_t = \hat{c} \quad P^x\text{-a.s.}, \quad (1.3)$$

where $\hat{b} = \frac{1}{1-p}$ and $\hat{c} = \left(\frac{\beta(1-p)^2}{p}\right)^{\hat{b}}$.

c) If $p = 1$ then

$$\lim_{t \rightarrow \infty} \frac{\log R_t}{\sqrt{t}} = \sqrt{2\beta} \quad P^x\text{-a.s.} \quad (1.4)$$

Note that Part a) of Theorem 1.2 is a special case of a result proved by Biggins [3, 4].

We can compare Theorems 1.1 and 1.2 for this branching random walk in an inhomogeneous branching potential with some analogous known results for Branching Brownian Motion. Consider a model for branching Brownian motion in an inhomogeneous potential where single particles move as standard Brownian motions, each branching into two new particles at instantaneous rate $\beta|x|^p$ when at position x , where $\beta > 0$, $p \geq 0$. This inhomogeneous BBM has been considered in Itô & McKean [9], Harris & Harris [8] and Berestycki et al. [1, 2] where, in particular, we find the following results:

Theorem 1.3 (Itô & McKean [9], Section 5.14.). *Consider a BBM in the potential $\beta|\cdot|^p$, $\beta > 0$, $p \geq 0$ started from $x \in \mathbb{R}$:*

a) If $p \leq 2$ then $T_{\text{explo}} = \infty$ P^x -a.s.

b) If $p > 2$ then $T_{\text{explo}} < \infty$ P^x -a.s.

Theorem 1.4 (Harris & Harris [8]). *Consider the BBM model with $\beta > 0$, $p \in [0, 2]$, $x \in \mathbb{R}$.*

a) If $p \in [0, 2)$ then

$$\lim_{t \rightarrow \infty} \frac{R_t}{t^{\hat{b}}} = \hat{a} \quad P^x\text{-a.s.} \quad (1.5)$$

where $\hat{b} = \frac{2}{2-p}$ and $\hat{a} = \left(\frac{\beta}{2}(2-p)^2\right)^{\frac{1}{2-p}}$.

b) If $p = 2$ then

$$\lim_{t \rightarrow \infty} \frac{\log R_t}{t} = \sqrt{2\beta} \quad P^x\text{-a.s.} \quad (1.6)$$

Comparing results, it can be seen that the inhomogeneous Branching Random Walk shows quite a different behaviour from the inhomogeneous Branching Brownian Motion, both in terms of the explosion criteria and regarding the asymptotic growth of the rightmost particle position.

We shall give a heuristic argument to help explain Theorems 1.1 - 1.4 in Section 2. The rest of the paper will then contain the detailed proofs of Theorems 1.1 and 1.2. In Section 3 we introduce a family of one-particle martingales. We also present some other relevant one-particle results, which will be used in later sections. Section 3 is self-contained and can be read out of the context of branching processes. In Section 4 we recall some standard techniques used in the analysis of branching systems, which include spines, additive martingales and martingale changes of measure. In Section 5 we prove Theorem 1.1 about the explosion time using standard spine methods. Section 6 is devoted to the proof of Theorem 1.2 about the rightmost particle using the spine methods again.

2 Heuristics

Theorems 1.1 - 1.4 are concerned with *almost sure* explosion and *almost sure* rightmost particle asymptotics. We can informally recover analogous *expectation* results with careful use of the well known Many-to-One Lemma (for example, see [7]), which reduces the expectation of the sum of functionals of particles alive at time t to the expectation of a single particle.

In particular, the expected number of particles alive at time t in the branching system is

$$\mathbb{E}^x |N_t| = \mathbb{E}^x e^{\int_0^t \beta(X_s) ds} = \mathbb{E}^x e^{\int_0^t \beta |X_s|^p ds} \quad (2.1)$$

where $(X_t)_{t \geq 0}$ is the single-particle process under \mathbb{P}^x . It is then relatively straightforward to check that if $(X_t)_{t \geq 0}$ is a Brownian motion, the expected number of particles at time t is: finite for all $t > 0$ if $p < 2$; finite for $t < \hat{t}$ and infinite for $t \geq \hat{t}$ for some constant \hat{t} when $p = 2$; and, infinite for all $t > 0$ if $p > 2$. Whereas, if $(X_t)_{t \geq 0}$ is a continuous-time random walk then the expected number of particles at time t is: finite for all $t > 0$ if $p < 1$; and, infinite for all $t > 0$ if $p > 1$. These computations give the critical value of p for explosion of the expected numbers of particles, and suggest the almost sure explosion criteria found in Theorems 1.1 and 1.3

The expected number of particles following 'close' to a given trajectory f up to time t is

$$\mathbb{E}^x \left(\sum_{u \in N_t} \mathbf{1}_{\{X_s^u \approx f(s) \ \forall s \in [0, t]\}} \right) = \mathbb{E}^x \left(\mathbf{1}_{\{X_s \approx f(s) \ \forall s \in [0, t]\}} e^{\int_0^t \beta |X_s|^p ds} \right). \quad (2.2)$$

If $(X_t)_{t \geq 0}$ is a continuous-time random walk then using heuristic methods which involve large deviations theory for Lévy processes (for example, see [6]), we find

$$\log \mathbb{E}^x \left(\mathbf{1}_{\{X_s \approx f(s) \ \forall s \in [0, t]\}} e^{\int_0^t \beta |X_s|^p ds} \right) \sim I_t(f) := \int_0^t \beta f(s)^p - \Lambda(f'(s)) ds,$$

where $\Lambda : [0, \infty) \rightarrow [0, \infty)$ is the rate function of the random walk given by

$$\Lambda(x) = 2\lambda + x \log \left(\frac{\sqrt{x^2 + 4\lambda^2} + x}{2\lambda} \right) - \sqrt{x^2 + 4\lambda^2} \sim x \log x \text{ as } x \rightarrow \infty.$$

(See Schilder's theorem for large deviations of paths in Brownian motion, where $\Lambda(x) = \frac{1}{2}x^2$.) Hence the expected number of particles following the curve f either grows exponentially or decays exponentially in t depending on the growth rate of f .

Further, we anticipate that the almost sure number of particles that have stayed close to path f over large time period $[0, t]$ will be roughly of order $\exp\{I_t(f)\}$ *as long as there have not been any extinction events along the path*, corresponding to the growth rate always remaining positive with $I_s(f) > 0$ for all $s \in (0, t]$. See Berestycki et al. [2] where such almost sure growth rates along paths are made rigorous for inhomogeneous BBM.

Thus, in order to find the almost sure asymptotic rightmost particle position, for t large we would like to find $\sup f(t)$ where the supremum is taken over all paths such that no extinction occurs, that is, over paths f with $I_s(f) > 0$ for all $s \in (0, t]$. In fact, it turns out that the optimal path f^* for the rightmost position then satisfies $I_s(f^*) = 0$ for all $s \in (0, t]$, that is, f^* solves the equation

$$\Lambda(f^{*'}(s)) = \beta f^*(s)^p.$$

Solving this equation for the inhomogeneous BRW leads exactly to the asymptotics of the rightmost particle as given in Theorem 1.2. Although we will not make the above heuristics rigorous for the BRW in this article, our more direct proof of Theorem 1.2, which we give in Section 6, will involve showing that there almost surely exists a particle staying close to the critical curve f^* .

3 Single-particle results

In this section we introduce a family of martingales for continuous-time random walks. Throughout this section the time set for all the processes is assumed to be $[0, T)$, where $T \in (0, \infty]$ is deterministic.

Suppose we are given a Poisson process $(Y_t)_{t \in [0, T)} \stackrel{d}{=} PP(\lambda)$ under a probability measure \mathbb{P} . Let us denote by J_i the time of the i^{th} jump of $(Y_t)_{t \in [0, T)}$. Then we have the following result.

Lemma 3.1. *Let $\theta : [0, T) \rightarrow [0, \infty)$ be a locally-integrable function. That is, $\int_0^t \theta(s) ds < \infty \forall t \in [0, T)$. Then the following process is a \mathbb{P} -martingale:*

$$M_t := e^{\int_0^t \log \theta(s) dY_s + \lambda \int_0^t (1 - \theta(s)) ds} = \left(\prod_{i: J_i \leq t} \theta(J_i) \right) e^{\lambda \int_0^t (1 - \theta(s)) ds}, \quad t \in [0, T),$$

where for any function f , $\int_0^t f(s) dY_s := \sum_{i: J_i \leq t} f(J_i)$.

The next result tells what effect the martingale $(M_t)_{t \in [0, T)}$ has on the process $(Y_t)_{t \in [0, T)}$ when used as a Radon-Nikodym derivative.

Lemma 3.2. *Let $(\hat{\mathcal{F}}_t)_{t \in [0, T)}$ be the natural filtration of $(Y_t)_{t \in [0, T)}$. Define the new measure \mathbb{Q} via*

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\hat{\mathcal{F}}_t} = M_t, \quad t \in [0, T).$$

Then under the new measure \mathbb{Q}

$$(Y_t)_{t \in [0, T)} \stackrel{d}{=} IPP(\lambda\theta(t)),$$

where $IPP(\lambda\theta(t))$ stands for time-inhomogeneous Poisson process of instantaneous jump rate $\lambda\theta(t)$.

Outline of the proof of Lemmas 3.1 and 3.2. As an intermediate step one can check by standard calculations that the following identity holds:

$$\mathbb{E} \left(e^{\int_0^t \log \theta(s) dY_s} \mathbf{1}_{\{Y_t = k\}} \right) = e^{-\lambda t} \frac{\lambda^k}{k!} \left(\int_0^t \theta(s) ds \right)^k \quad \forall k \in \mathbb{N}, \quad (3.1)$$

where \mathbb{E} is the expectation associated with \mathbb{P} .

The martingale property of $(M_t)_{t \in [0, T)}$ then follows immediately.

To verify that under \mathbb{Q} , $(Y_t)_{t \in [0, T)}$ is a time-inhomogeneous Poisson process one can check the finite-dimensional distributions. \square

For the next few results suppose that $(Y_t)_{t \in [0, T)} \stackrel{d}{=} IPP(r(t))$, where $r : [0, T) \rightarrow [0, \infty)$ is a locally-integrable function. That is, $(Y_t)_{t \in [0, T)}$ is a time-inhomogeneous Poisson process with instantaneous jump rate $r(t)$.

The following identity is a standard integration by-parts-formula which is trivial to prove.

Proposition 3.3 (Integration by parts for time-inhomogeneous Poisson processes). *Let $f \in C^1([0, T))$. Then almost surely*

$$\int_0^t f(s) dY_s = f(t)Y_t - \int_0^t f'(s)Y_s ds,$$

Since $(Y_t)_{t \in [0, T]} \stackrel{d}{=} (Z_{R(t)})_{t \in [0, T]}$, where $R(t) := \int_0^t r(s)ds$ and $(Z_t)_{t \geq 0} \stackrel{d}{=} PP(1)$ we also have the following useful result.

Proposition 3.4 (SLLN for time-inhomogeneous Poisson processes).

If $\lim_{t \rightarrow T} \int_0^t r(s)ds = \infty$ then

$$\frac{Y_t}{\int_0^t r(s)ds} \rightarrow 1 \text{ a.s. as } t \rightarrow T.$$

The next result combines Propositions 3.3 and 3.4.

Proposition 3.5. Let $f : [0, T] \rightarrow [0, \infty)$ be differentiable such that $f'(t) \geq 0$ for t large enough. Suppose r and f satisfy the following two conditions:

$$(i) \int_0^t r(s)ds \rightarrow \infty \text{ as } t \rightarrow T$$

$$(ii) \limsup_{t \rightarrow T} \frac{f(t) \int_0^t r(s)ds}{\int_0^t f(s)r(s)ds} < \infty$$

Then

$$\frac{\int_0^t f(s)dY_s}{\int_0^t f(s)r(s)ds} \rightarrow 1 \text{ a.s. as } t \rightarrow T.$$

Note that the second condition is generally rather restrictive, but it will be satisfied by the functions that we consider in this article.

Proof. Observe that by Proposition 3.3 we have

$$\frac{\int_0^t f(s)dY_s}{\int_0^t f(s)r(s)ds} = \frac{f(t)Y_t - \int_0^t f'(s)Y_s ds}{\int_0^t f(s)r(s)ds}.$$

Then apply Proposition 3.4 and use the deterministic integration-by-parts formula. \square

Let us now consider a continuous-time random walk $(X_t)_{t \in [0, T]}$ defined under some probability measure \mathbb{P} as it was described in the introduction. It can be written as a difference of two independent Poisson processes of rate λ :

$$X_t = X_t^+ - X_t^-, \quad t \in [0, T],$$

where $(X_t^+)_{t \in [0, T]}$ is the process of positive jumps and $(X_t^-)_{t \in [0, T]} \stackrel{d}{=} PP(\lambda)$ is the process of negative jumps. From Lemmas 3.1 and 3.2 we get the following result.

Proposition 3.6. Let $\theta^+, \theta^- : [0, T] \rightarrow [0, \infty)$ be two locally-integrable functions. Then the following process is a \mathbb{P} -martingale:

$$M_t := e^{\int_0^t \log \theta^+(s) dX_s^+ + \lambda \int_0^t (1 - \theta^+(s)) ds + \int_0^t \log \theta^-(s) dX_s^- + \lambda \int_0^t (1 - \theta^-(s)) ds}, \quad t \in [0, T]. \quad (3.2)$$

Moreover, if we define the new measure \mathbb{Q} as

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\hat{\mathcal{F}}_t} = M_t, \quad t \in [0, T],$$

where $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$ is the natural filtration of $(X_t)_{t \in [0, T]}$, then under \mathbb{Q}

$$(X_t^+)_{t \in [0, T]} \stackrel{d}{=} IPP(\lambda \theta^+(t)), \quad (X_t^-)_{t \in [0, T]} \stackrel{d}{=} IPP(\lambda \theta^-(t)).$$

In other words the martingale M used as the Radon-Nikodym derivative has the effect of scaling the upward jumps by the factor of $\theta^+(t)$ and the rate of downward jumps by the factor $\theta^-(t)$ at time t .

Furthermore from Propositions 3.4 and 3.5 we know that \mathbb{Q} -a.s.

$$\lim_{t \rightarrow T} \frac{X_t^+}{\int_0^t \lambda \theta^+(s) ds} = 1, \quad \lim_{t \rightarrow T} \frac{X_t^-}{\int_0^t \lambda \theta^-(s) ds} = 1,$$

$$\lim_{t \rightarrow T} \frac{\int_0^t f(s) dX_s^+}{\int_0^t \lambda \theta^+(s) f(s) ds} = 1, \quad \lim_{t \rightarrow T} \frac{\int_0^t f(s) dX_s^-}{\int_0^t \lambda \theta^-(s) f(s) ds} = 1$$

provided that θ^+ , θ^- and f satisfy the conditions of Propositions 3.4 and 3.5.

4 Spines and additive martingales

In this section we give a brief overview of the main spine tools. The major reference for this section is the work of Hardy and Harris [7] where all the proofs and further references can be found.

Firstly, let us take the time set of our model to be $[0, T)$ for some deterministic $T \in (0, \infty]$. We assume in this section that the branching process starts from 0.

We let $(\mathcal{F}_t)_{t \in [0, T)}$ denote the natural filtration of our branching process as described in the introduction. We define $\mathcal{F}_T := \sigma(\cup_{t \in [0, T)} \mathcal{F}_t)$.

Let us now extend our branching random walk by identifying an infinite line of descent, which we refer to as the spine, in the following way. The initial particle of the branching process begins the spine. When it splits into two new particle, one of them is chosen with probability $\frac{1}{2}$ to continue the spine. This goes on in the obvious way: whenever the particle currently in the spine splits, one of its children is chosen uniformly at random to continue the spine.

The spine is denoted by $\xi = \{\emptyset, \xi_1, \xi_2, \dots\}$, where \emptyset is the initial particle (both in the spine and in the entire branching process) and ξ_n is the particle in the $(n+1)^{st}$ generation of the spine. Furthermore, at time $t \in [0, T)$ we define:

- $node_t(\xi) := u \in N_t \cap \xi$ (such u is unique). That is, $node_t(\xi)$ is the particle in the spine alive at time t .
- $n_t := |node_t(\xi)|$. Thus n_t is the number of fissions that have occurred along the spine by time t .
- $\xi_t := X_t^u$ for $u \in N_t \cap \xi$. So $(\xi_t)_{t \in [0, T)}$ is the path of the spine.

The next important step is to define a number of filtrations of our sample space, which contain different information about the process.

Definition 4.1 (Filtrations).

- \mathcal{F}_t was defined earlier. It is the filtration which knows everything about the particles' motion and their genealogy, but it knows nothing about the spine.
- We also define $\tilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t, node_t(\xi))$. Thus $\tilde{\mathcal{F}}$ has all the information about the process together with all the information about the spine. This will be the largest filtration.
- $\mathcal{G}_t := \sigma(\xi_s : 0 \leq s \leq t)$. This filtration only has information about the path of the spine process, but it can't tell which particle $u \in N_t$ is the spine particle at time t .

- $\tilde{\mathcal{G}}_t := \sigma(\mathcal{G}_t, (\text{node}_s(\xi) : 0 \leq s \leq t))$. This filtration knows everything about the spine including which particles make up the spine, but it doesn't know what is happening off the spine.

Note that $\mathcal{G}_t \subset \tilde{\mathcal{G}}_t \subset \tilde{\mathcal{F}}_t$ and $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$. We shall be using these filtrations throughout the whole article for taking various conditional expectations.

We let \tilde{P} be the probability measure under which the branching random walk is defined together with the spine. Hence $P = \tilde{P}|_{\mathcal{F}_T}$. We shall write \tilde{E} for the expectation with respect to \tilde{P} .

Under \tilde{P} the entire branching process (with the spine) can be described in the following way.

- the initial particle (the spine) moves like a random walk.
- At instantaneous rate $\beta|\cdot|^p$ it splits into two new particles.
- One of these particles (chosen uniformly at random) continues the spine. That is, it continues moving as a random walk and branching at rate $\beta|\cdot|^p$.
- The other particle initiates a new independent P -branching processes from the position of the split

It is not hard to see that under \tilde{P} the spine's path $(\xi_t)_{t \in [0, T]}$ is itself a continuous-time random walk.

Also, conditional on the path of the spine, $(n_t)_{t \in [0, T]}$ is a time-inhomogeneous Poisson process (or a Cox process) with instantaneous jump rate $\beta|\xi_t|^p$. That is, conditional on \mathcal{G}_t , k splits take place along the spine by time t with probability

$$\tilde{P}(n_t = k | \mathcal{G}_t) = \frac{(\int_0^t \beta |\xi_s|^p ds)^k}{k!} e^{-\int_0^t \beta |\xi_s|^p ds}.$$

The next result (see e.g. [7]) has already been mentioned in the introduction.

Theorem 4.2 (Many-to-One Theorem). *Let $f(t) \in m\mathcal{G}_t$. In other words, $f(t)$ is \mathcal{G}_t -measurable. Suppose it has the representation*

$$f(t) = \sum_{u \in N_t} f_u(t) \mathbf{1}_{\{\text{node}_t(\xi) = u\}},$$

where $f_u(t) \in m\mathcal{F}_t$, then

$$E\left(\sum_{u \in N_t} f_u(t)\right) = \tilde{E}\left(f(t) e^{\int_0^t \beta |\xi_s|^p ds}\right).$$

Now let $\theta = (\theta^+, \theta^-)$, where $\theta^+, \theta^- : [0, T] \rightarrow [0, \infty)$ are two locally-integrable functions. In view of Proposition 3.6 we define the following \tilde{P} -martingale w.r.t filtration $(\tilde{\mathcal{G}}_t)_{t \in [0, T]}$:

$$\begin{aligned} \tilde{M}_\theta(t) := & e^{-\beta \int_0^t |\xi_s|^p ds} 2^{n_t} \times \exp\left(\int_0^t \log \theta^+(s) d\xi_s^+ + \int_0^t \lambda(1 - \theta^+(s)) ds \right. \\ & \left. + \int_0^t \log \theta^-(s) d\xi_s^- + \int_0^t \lambda(1 - \theta^-(s)) ds\right), \end{aligned} \quad (4.1)$$

where $(\xi_t^+)_{t \in [0, T]}$ is the process of positive jumps of the spine process and $(\xi_t^-)_{t \in [0, T]}$ is the process of its negative jumps.

Note that \tilde{M}_θ is the product of two \tilde{P} -martingales, the first of which doubles the branching rate along the spine, and the second biases the rates of upward and downward jumps of the spine process. If we define the probability measure \tilde{Q}_θ as

$$\left. \frac{d\tilde{Q}_\theta}{d\tilde{P}} \right|_{\tilde{\mathcal{F}}_t} = \tilde{M}_\theta(t), \quad t \in [0, T) \quad (4.2)$$

then under \tilde{Q}_θ the branching process has the following description:

Proposition 4.3 (Branching process under \tilde{Q}_θ).

- The initial particle (the spine) moves like a biased random walk. That is, at time t it jumps up at instantaneous rate $\lambda\theta^+(t)$ and jumps down at instantaneous rate $\lambda\theta^-(t)$.
- When it is at position x it splits into two new particles at instantaneous rate $2\beta|x|^p$.
- One of these particles (chosen uniformly at random) continues the spine. I.e. it continues moving as a biased random walk and branching at rate $2\beta|\cdot|^p$.
- The other particle initiates an unbiased branching process (as under P) from the position of the split.

Note that although (4.2) only defines \tilde{Q}_θ on events in $\cup_{t \in [0, T)} \tilde{\mathcal{F}}_t$, Carathéodory's extension theorem tells that \tilde{Q}_θ has a unique extension on $\tilde{\mathcal{F}}_T := \sigma(\cup_{t \in [0, T)} \tilde{\mathcal{F}}_t)$ and thus (4.2) implicitly defines \tilde{Q}_θ on $\tilde{\mathcal{F}}_T$.

Proposition 4.4 (Additive martingale). We define the probability measure $Q_\theta := \tilde{Q}_\theta|_{\mathcal{F}_T}$ so that

$$\left. \frac{dQ_\theta}{dP} \right|_{\mathcal{F}_t} = M_\theta(t), \quad t \in [0, T), \quad (4.3)$$

where $M_\theta(t)$ is the additive martingale

$$\begin{aligned} M_\theta(t) = & \sum_{u \in N_t} \exp \left(\int_0^t \log \theta^+(s) dX_u^+(s) + \int_0^t \log \theta^-(s) dX_u^-(s) \right. \\ & \left. + \int_0^t \lambda(2 - \theta^+(s) - \theta^-(s)) ds - \beta \int_0^t |X_u(s)|^p ds \right) \end{aligned} \quad (4.4)$$

and $(X_u^+(s))_{0 \leq s \leq t}$ is the process of positive jumps of particle u , $(X_u^-(s))_{0 \leq s \leq t}$ is the process of its negative jumps.

Let us recall the following measure-theoretic result, which gives Lebesgue's decomposition of Q_θ into absolutely-continuous and singular parts. It can for example be found in the book of R. Durrett [5] (Section 4.3).

Lemma 4.5. For events $A \in \mathcal{F}_T$

$$Q_\theta(A) = \int_A \limsup_{t \rightarrow T} M_\theta(t) dP + Q_\theta(A \cap \{\limsup_{t \rightarrow T} M_\theta(t) = \infty\}). \quad (4.5)$$

In view of this lemma one will be interested in identifying the set of values of θ for which $\limsup_{t \rightarrow T} M_\theta(t) < \infty$ Q_θ -a.s., in which case $Q_\theta \ll P$ on \mathcal{F}_T . An important tool for doing this is the so-called spine decomposition.

Lemma 4.6 (Spine decomposition).

$$\begin{aligned}
E^{\tilde{Q}_\theta} \left(M_\theta(t) | \tilde{\mathcal{G}}_T \right) &= \exp \left(\int_0^t \log \theta^+(s) d\xi_s^+ + \int_0^t \log \theta^-(s) d\xi_s^- \right. \\
&\quad \left. + \lambda \int_0^t (2 - \theta^+(s) - \theta^-(s)) ds - \beta \int_0^t |\xi_s|^p ds \right) \\
&\quad + \sum_{u < \text{node}_t(\xi)} \exp \left(\int_0^{S_u} \log \theta^+(s) d\xi_s^+ + \int_0^{S_u} \log \theta^-(s) d\xi_s^- \right. \\
&\quad \left. + \lambda \int_0^{S_u} (2 - \theta^+(s) - \theta^-(s)) ds - \beta \int_0^{S_u} |\xi_s|^p ds \right), \quad (4.6)
\end{aligned}$$

where $\{S_u : u \in \xi\}$ is the set of fission times along the spine.

The first term is called the *spine term* or *spine(t)* and the second one is called the *sum term* or *sum(t)*.

5 Explosion: proof of Theorem 1.1

5.1 Case $p \leq 1$

Firstly, we shall prove that $T_{\text{explo}} = \infty$ P^x -a.s. if the exponent of the branching rate p is ≤ 1 . As in the proof of Theorem 1.3 a) from [9] for the BBM model it will be sufficient to show that $E|N_t| < \infty$ for some $t > 0$ as it is explained below.

Let us begin with the simple observation, which says that the starting position of the branching process is not important in Theorem 1.1. Thus we shall take it to be 0 in the rest of this section.

Proposition 5.1.

$$P^x(T_{\text{explo}} = \infty) = P^y(T_{\text{explo}} = \infty) \quad \forall x, y \in \mathbb{Z}.$$

Proof. Take any x and $y \in \mathbb{Z}$ and start a branching random walk from x . Let T_y be the first passage time of the process to level y . That is,

$$T_y := \inf\{t : \exists u \in N_t \text{ s.t. } X_t^u = y\}.$$

$T_y < \infty$ because a random walk started from any level x will hit any level y . Then by the strong Markov property of the branching process the subtree initiated from y at time T_y has the same law as a branching random walk started from y . Consequently, if the explosion does not happen in the big tree started from x , it cannot happen in its subtree started from y . Thus

$$P^x(T_{\text{explo}} = \infty) \leq P^y(T_{\text{explo}} = \infty).$$

Since x and y were arbitrary it follows that

$$P^x(T_{\text{explo}} = \infty) = P^y(T_{\text{explo}} = \infty) \quad \forall x, y \in \mathbb{Z}.$$

□

One important corollary to the previous result is the following 0-1 law.

Corollary 5.2.

$$P(T_{\text{explo}} = \infty) \in \{0, 1\}.$$

Proof. If X_1 is the position of the first split then from the branching property we have

$$P(T_{\text{explo}} = \infty) = E\left((P^{X_1}(T_{\text{explo}} = \infty))^2\right) = (P(T_{\text{explo}} = \infty))^2.$$

Thus $P(T_{\text{explo}} = \infty) \in \{0, 1\}$. □

Let us now state another useful fact.

Proposition 5.3. *Take some deterministic time $t > 0$.*

$$\text{If } P(T_{\text{explo}} < t) = 0 \text{ then } P^x(T_{\text{explo}} < t) = 0 \text{ } \forall x \in \mathbb{Z}.$$

Proof. Consider a branching process started from 0. Take any $\epsilon \in (0, t)$. Let T_x be the hitting time of level x as in Proposition 5.1. Then there is a positive probability that the process will hit level x before time ϵ . Then

$$\begin{aligned} 0 &= P(T_{\text{explo}} < t) \geq P(T_{\text{explo}} < t, T_x < \epsilon) \geq P(T_{\text{explo}}^x < t - \epsilon, T_x < \epsilon) \\ &= E\left(P(T_{\text{explo}}^x < t - \epsilon, T_x < \epsilon | T_x)\right) = P(T_x < \epsilon) P^x(T_{\text{explo}} < t - \epsilon), \end{aligned}$$

where T_{explo}^x is the explosion time of the subtree started from x . Thus, since $P(T_x < \epsilon) > 0$ we find that

$$P^x(T_{\text{explo}} < t - \epsilon) = 0.$$

Since ϵ was arbitrary, letting $\epsilon \downarrow 0$ gives the result. □

As a consequence of Proposition 5.3 we get the following corollary.

Corollary 5.4. *Let $t > 0$ be any deterministic time.*

$$\text{if } P(T_{\text{explo}} \geq t) = 1 \text{ then } P(T_{\text{explo}} = \infty) = 1.$$

In particular, if $E|N_t| < \infty$ then $P(T_{\text{explo}} < \infty) = 0$.

Proof. The result follows by induction since if the original tree almost surely does not explode by time t then none of its subtrees initiated at time t will explode by time $2t$ and one can repeat this argument any number of times. □

Proof of Theorem 1.1 a). We wish to show that if $p \leq 1$ then $P(T_{\text{explo}} = \infty) = 1$. From Corollary 5.4, it is sufficient to show that $E(|N_t|) < \infty$ for some $t > 0$.

By the Many-to-One Theorem (Theorem 4.2)

$$E(|N_t|) = E\left(\sum_{u \in N_t} 1\right) = \tilde{E}\left(e^{\int_0^t \beta |\xi_s|^p ds}\right),$$

where $(\xi_t)_{t \geq 0}$ is a continuous-time random walk under \tilde{P} . Recall, $\xi_t = \xi_t^+ - \xi_t^-$, where $(\xi_t^+)_{t \geq 0}$ and $(\xi_t^-)_{t \geq 0}$ are two independent Poisson processes with jump rate λ . Then

$$\begin{aligned} \tilde{E}\left(e^{\int_0^t \beta |\xi_s|^p ds}\right) &\leq \tilde{E}\left(e^{t\beta \sup_{0 \leq s \leq t} |\xi_s|^p}\right) \\ &= \tilde{E}\left(e^{t\beta \sup_{0 \leq s \leq t} |\xi_s^+ - \xi_s^-|^p}\right) \leq \tilde{E}\left(e^{t\beta \sup_{0 \leq s \leq t} ((\xi_s^+)^p \vee (\xi_s^-)^p)}\right) \\ &= \tilde{E}\left(e^{t\beta ((\xi_t^+)^p \vee (\xi_t^-)^p)}\right) \leq \tilde{E}\left(e^{t\beta ((\xi_t^+)^p + (\xi_t^-)^p)}\right) \\ &= \left[\tilde{E}\left(e^{t\beta (\xi_t^+)^p}\right)\right]^2 \leq \left[\tilde{E}\left(e^{t\beta \xi_t^+}\right)\right]^2 \end{aligned}$$

since ξ^+ is supported on $\{0, 1, 2, \dots\}$ whence $(\xi_t^+)^p \leq \xi_t^+$ for $p \in [0, 1]$. Then

$$\tilde{E}\left(e^{t\beta\xi_t^+}\right) = \sum_{n=0}^{\infty} e^{\beta tn} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \exp\{e^{\beta t}\lambda t - \lambda t\} < \infty \quad \forall t \geq 0.$$

Thus $E(|N_t|) < \infty$ for all $t > 0$. □

5.2 Case $p > 1$

Proof of Theorem 1.1 b). We wish to show that if $p > 1$ then $P(T_{\text{explo}} < \infty) = 1$. By Corollary 5.2 this is equivalent to $P(T_{\text{explo}} < \infty) > 0$. It would be sufficient to prove that $P(T_{\text{explo}} \leq T) > 0$ for all $T > 0$. For a contradiction we suppose that there exists $T > 0$ s.t.

$$P(T_{\text{explo}} \leq T) = 0. \tag{5.1}$$

We fix this T for the rest of this subsection. Under the assumption (5.1) that there is no explosion before time T we can perform the usual spine construction on $[0, T]$. The key steps of the proof can then be summarised as follows:

- (i) We choose $\theta^+, \theta^- : [0, T] \rightarrow [0, \infty)$ such that at time T
 - (A) the spine process ξ_t goes to ∞ under \tilde{Q}_θ
 - (B) the additive martingale M_θ from satisfies $\limsup_{t \rightarrow T} M_\theta(t) < \infty$ Q_θ -a.s.
- (ii) We deduce that $Q_\theta \ll P$ on \mathcal{F}_T , whence with positive P -probability one particle goes to ∞ at time T giving infinitely many births along its path.
- (iii) We get a contradiction to (5.1).

We take $\theta^-(\cdot) \equiv 1$. That is, we leave the negative jumps of the spine process unaltered under \tilde{Q}_θ . $\theta^+(\cdot)$ needs to be chosen carefully such that both (A) and (B) above are satisfied. One such choice is

$$\theta^+(s) = (T - s)^{-c}, \quad s \in [0, T], \tag{5.2}$$

where $c > \frac{p}{p-1}$ (e.g. take $c = \frac{p}{p-1} + 1$).

The additive martingale (4.4) in this case takes the following form (with $\theta^+(\cdot)$ defined above)

$$M_\theta(t) = \sum_{u \in N_t} \exp\left(\int_0^t \log \theta^+(s) dX_u^+(s) + \int_0^t \lambda(1 - \theta^+(s)) ds - \beta \int_0^t |X_u(s)|^p ds\right), \quad t \in [0, T], \tag{5.3}$$

If we can now show that

$$\limsup_{t \rightarrow T} M_\theta(t) < \infty \quad Q_\theta\text{-a.s.} \tag{5.4}$$

it would follow from Lemma 4.5 that $Q_\theta \ll P$ on \mathcal{F}_T .

To prove (5.4) it is sufficient to show that

$$\limsup_{t \rightarrow T} E^{\tilde{Q}_\theta}\left(M_\theta(t) | \tilde{\mathcal{G}}_T\right) < \infty \quad \tilde{Q}_\theta\text{-a.s.}, \tag{5.5}$$

since if (5.5) holds then by Fatou's lemma

$$\begin{aligned} E^{\tilde{Q}_\theta} \left(\liminf_{t \rightarrow T} M_\theta(t) \middle| \tilde{\mathcal{G}}_T \right) &\leq \liminf_{t \rightarrow T} E^{\tilde{Q}_\theta} \left(M_\theta(t) \middle| \tilde{\mathcal{G}}_T \right) \\ &\leq \limsup_{t \rightarrow T} E^{\tilde{Q}_\theta} \left(M_\theta(t) \middle| \tilde{\mathcal{G}}_T \right) < \infty \quad \tilde{Q}_\theta\text{-a.s.}, \end{aligned}$$

therefore $\liminf_{t \rightarrow T} M_\theta(t) < \infty$ \tilde{Q}_θ -a.s. and hence also Q_θ -a.s. Then since $\frac{1}{M_\theta(t)}$ is a positive Q_θ -supermartingale on $[0, T)$, it must converge Q_θ -a.s., hence

$$\limsup_{t \rightarrow T} M_\theta(t) = \liminf_{t \rightarrow T} M_\theta(t) < \infty \quad Q_\theta\text{-a.s.}$$

So let us now prove (5.5). Recall the spine decomposition (4.6):

$$E^{\tilde{Q}_\theta} \left(M_\theta(t) \middle| \tilde{\mathcal{G}}_T \right) = \text{spine}(t) + \text{sum}(t),$$

where

$$\text{spine}(t) = \exp \left(\int_0^t \log \theta^+(s) d\xi_s^+ + \int_0^t \lambda(1 - \theta^+(s)) ds - \int_0^t \beta |\xi_s|^p ds \right)$$

and

$$\text{sum}(t) = \sum_{u < \text{node}_t(\xi)} \text{spine}(S_u).$$

We start by proving the following assertion about the spine term.

Proposition 5.5. *There exist some \tilde{Q}_θ -a.s. finite positive random variables C' , C'' and a random time $T' \in [0, T)$ such that $\forall t > T'$*

$$\text{spine}(t) \leq C' \exp \left(-C''(T-t)^{-p(c-1)+1} \right).$$

Proof of Proposition 5.5. From Proposition 3.6 under \tilde{Q}_θ the process $(\xi_t^+)_{t \in [0, T)}$ is a time-inhomogeneous Poisson process of rate $\lambda \theta^+(t)$ and $(\xi_t^-)_{t \in [0, T)}$ is a Poisson process of rate λ .

Using the standard integration-by-parts formula one can check that

$$\int_0^t (T-s)^{-c} \log(T-s) ds \sim \frac{1}{c-1} (T-t)^{-c+1} \log(T-t) \text{ as } t \rightarrow T.$$

Hence for θ^+ defined as in (5.2)

$$\limsup_{t \rightarrow T} \frac{\log \theta^+(t) \int_0^t \lambda \theta^+(s) ds}{\int_0^t \log \theta^+(s) \lambda \theta^+(s) ds} = 1.$$

Also $\int_0^t \lambda \theta^+(s) ds = \lambda(c-1)^{-1} (T-t)^{-c+1} \rightarrow \infty$ as $t \rightarrow T$ and $\log \theta^+(\cdot)$ is increasing. Thus from Proposition 3.4 and Proposition 3.5 we have that \tilde{Q}_θ -a.s.

$$\frac{\xi_t}{\int_0^t \lambda \theta^+(s) ds} \rightarrow 1, \quad \frac{\int_0^t \log \theta^+(s) d\xi_s^+}{\int_0^t \log \theta^+(s) \lambda \theta^+(s) ds} \rightarrow 1.$$

Combining these observations we get that $\forall \epsilon > 0 \exists \tilde{Q}_\theta$ -a.s. finite time T_ϵ such that $\forall t > T_\epsilon$ the following inequalities are true:

$$\begin{aligned} \int_0^t \log \theta^+(s) d\xi_s^+ &< (1 + \epsilon) \int_0^t \log \theta^+(s) \lambda \theta^+(s) ds \\ &= (1 + \epsilon) \int_0^t -c \log(T - s) \lambda (T - s)^{-c} ds; \end{aligned}$$

$$\begin{aligned} |\xi_t| &> (1 - \epsilon) \int_0^t \lambda \theta^+(s) ds = (1 - \epsilon) \frac{\lambda}{c - 1} (T - t)^{-c+1}; \\ \lambda(1 - \theta^+(t)) &< 0; \\ \log(T - t) \lambda (T - t)^{-c} &\leq \frac{1}{2} \frac{\beta \left(\frac{\lambda}{c-1} (1 - \epsilon) \right)^p}{\lambda c (1 + \epsilon)} (T - t)^{-(c-1)p}. \end{aligned}$$

Thus, for $t > T_\epsilon$ we have

$$\begin{aligned} spine(t) &= \exp \left(\int_0^t \log \theta^+(s) d\xi_s^+ + \int_0^t \lambda(1 - \theta^+(s)) ds - \int_0^t \beta |\xi_s|^p ds \right) \\ &\leq C_\epsilon \exp \left\{ (1 + \epsilon) \int_0^t -c \lambda \log(T - s) (T - s)^{-c} ds \right. \\ &\quad \left. - \beta \int_0^t \left(\frac{\lambda(1 - \epsilon)}{c - 1} (T - s)^{-c+1} \right)^p ds \right\} \\ &\leq C'_\epsilon \exp \left\{ -\frac{1}{2} \beta \left(\frac{\lambda(1 - \epsilon)}{c - 1} \right)^p \frac{1}{p(c - 1) - 1} (T - t)^{-(c-1)p+1} \right\}, \end{aligned}$$

where C_ϵ and C'_ϵ are some \tilde{Q}_θ -a.s. finite random variables, which don't depend on t . Letting $T' = T_\epsilon$, $C' = C'_\epsilon$ and $C'' = \frac{1}{2} \beta \left(\frac{\lambda(1 - \epsilon)}{c - 1} \right)^p \frac{1}{p(c - 1) - 1}$ we finish the proof of Proposition 5.5. \square

We now look at the *sum* term:

$$\begin{aligned} sum(t) &= \sum_{u < node_t(\xi)} spine(S_u) \\ &= \left(\sum_{u < node_t(\xi), S_u \leq T'} spine(S_u) \right) + \left(\sum_{u < node_t(\xi), S_u > T'} spine(S_u) \right) \\ &\leq \sum_{u < node_t(\xi), S_u \leq T'} spine(S_u) \\ &\quad + \sum_{u < node_t(\xi), S_u > T'} C' \exp \left(-C'' (T - S_u)^{-p(c-1)+1} \right) \end{aligned}$$

using Proposition 5.5. The first sum is \tilde{Q}_θ -a.s. bounded since it only counts births up to time T' . Call an upper bound on the first sum C_1 . Then we have

$$sum(t) \leq C_1 + C' \sum_{n=1}^{\infty} \exp \left(-C'' (T - S_n)^{-p(c-1)+1} \right), \quad (5.6)$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t \in [0, T]}$ conditional on the path of the spine is time-inhomogeneous Poisson process (or Cox process) with birth rate $2\beta|\xi_t|^p$ at time t . Thus as $t \rightarrow T$, almost surely under \tilde{Q}_θ

$$n_t \sim \int_0^t 2\beta|\xi_s|^p ds \sim 2\beta\left(\frac{\lambda}{c-1}\right)^p \frac{1}{p(c-1)-1} (T-t)^{-p(c-1)+1}, \quad (5.7)$$

hence,

$$n \sim 2\beta\left(\frac{\lambda}{c-1}\right)^p \frac{1}{p(c-1)-1} (T-S_n)^{-p(c-1)+1}.$$

So for some \tilde{Q}_θ -a.s. finite positive random variable C_2 we have

$$(T-S_n)^{-p(c-1)+1} \geq C_2 n \quad \forall n.$$

Then substituting this into (5.6) we get

$$sum(t) \leq C_1 + C' \sum_{n=1}^{\infty} e^{-C'' C_2 n},$$

which is bounded \tilde{Q}_θ -a.s. We have thus shown that

$$\limsup_{t \rightarrow T} E^{\tilde{Q}_\theta} \left(M_\theta(t) | \tilde{\mathcal{G}}_T \right) = \limsup_{t \rightarrow T} \left(spine(t) + sum(t) \right) < \infty \quad \tilde{Q}_\theta\text{-a.s.}$$

proving (5.5) and consequently (5.4).

From Lemma 4.5 it now follows that for events $A \in \mathcal{F}_T$

$$Q_\theta(A) = \int_A \limsup_{t \rightarrow T} M_\theta(t) dP.$$

Thus $Q_\theta(A) > 0 \Rightarrow P(A) > 0$. Let us consider the event $\{|N_t| \rightarrow \infty \text{ as } t \rightarrow T\}$. From (5.7) we have $\tilde{Q}_\theta(n_t \rightarrow \infty \text{ as } t \rightarrow T) = 1$, so $Q_\theta(|N_t| \rightarrow \infty \text{ as } t \rightarrow T) = 1$ and then $P(|N_t| \rightarrow \infty \text{ as } t \rightarrow T) > 0$. Thus $P(T_{explo} \leq T) > 0$, which contradicts the initial assumption (5.1). Therefore, $P(T_{explo} \leq T) > 0, \forall T > 0$ and hence by Corollary 5.2

$$T_{explo} < \infty \text{ } P\text{-a.s.}$$

This completes the proof of Theorem 1.1 □

6 The rightmost particle: proof of Theorem 1.2

In this section we consider a branching random walk in the potential $\beta|\cdot|^p$, $\beta > 0$, $p \in [0, 1]$. By Theorem 1.1 there is no explosion of the population and so we take the time set of the branching process to be $[0, \infty)$. That is, in the set-up presented in Section 4 we let $T = \infty$.

Just like with the explosion probability in Section 5, the starting position of the branching process does not affect the behaviour of the rightmost particle in Theorem 1.2. For example in part a) suppose we know that $P^x(\lim_{t \rightarrow \infty} t^{-1} R_t = \lambda(\hat{\theta} - \hat{\theta}^{-1})) = 1$ for some $x \in \mathbb{Z}$. Take some $y \in \mathbb{Z}$. Then a branching process started from x will contain a subtree started from y . Hence $P^y(\limsup_{t \rightarrow \infty} t^{-1} R_t \leq \lambda(\hat{\theta} - \hat{\theta}^{-1})) = 1$. Also a branching process started from y will contain a subtree started from x . Hence $P^y(\liminf_{t \rightarrow \infty} t^{-1} R_t \geq \lambda(\hat{\theta} - \hat{\theta}^{-1})) = 1$ and so $P^y(\lim_{t \rightarrow \infty} t^{-1} R_t = \lambda(\hat{\theta} - \hat{\theta}^{-1})) = 1$. We shall thus take the starting position of the branching process to be 0 in the forthcoming proof presented in Subsections 6.1 - 6.3.

Our proof follows a similar approach as was used for the BBM model in J. Harris and S. Harris in [8].

6.1 Convergence properties of M_θ (under Q_θ)

We let M_θ be the additive martingale as defined in (4.4) for a given parameter θ . Note that since each M_θ is a positive P -martingale it must converge P -almost surely to a finite limit $M_\theta(\infty)$. We are interested in those values of θ for which $M_\theta(\infty)$ is strictly positive. The following result deals with this question.

Theorem 6.1.

Case A ($p = 0$), homogeneous branching:

Recall $\hat{\theta}$ from (1.2) which solves (uniquely)

$$\left(\theta - \frac{1}{\theta}\right) \log \theta - \left(\theta + \frac{1}{\theta}\right) + 2 = \frac{\beta}{\lambda} \quad \text{on } (1, \infty)$$

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_0$ and $\theta^-(\cdot) \equiv \frac{1}{\theta_0}$ for some constant $\theta_0 > 1$. Then

i) $\theta_0 < \hat{\theta} \Rightarrow M_\theta$ is UI and $M_\theta(\infty) > 0$ P -a.s. (under P).

ii) $\theta_0 > \hat{\theta} \Rightarrow M_\theta(\infty) = 0$ P -a.s.

Case B ($p \in (0, 1)$), inhomogeneous subcritical branching:

$$\text{Let } \hat{b} = \frac{1}{1-p}, \quad \hat{c} = \left(\frac{\beta(1-p)^2}{p}\right)^{\hat{b}} \text{ as in (1.3).}$$

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, and for a given $c > 0$,

$$\theta^+(s) := \frac{c}{\lambda(1-p)} \frac{s^{\hat{b}-1}}{(\log(s+2))^{\hat{b}}}, \quad s \geq 0.$$

Then

i) $c < \hat{c} \Rightarrow M_\theta$ is UI and $M_\theta(\infty) > 0$ P -a.s.

ii) $c > \hat{c} \Rightarrow M_\theta(\infty) = 0$ P -a.s.

Case C ($p = 1$), inhomogeneous near-critical branching:

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, and for a given $\alpha > 0$,

$$\theta^+(s) := e^{\alpha\sqrt{s}}, \quad s \geq 0.$$

Then

i) $\alpha < \sqrt{2\beta} \Rightarrow M_\theta$ is UI and $M_\theta(\infty) > 0$ P -a.s.

ii) $\alpha > \sqrt{2\beta} \Rightarrow M_\theta(\infty) = 0$ P -a.s.

The importance of this Theorem comes from the fact that if M_θ is P -uniformly integrable and $M_\theta(\infty) > 0$ P -a.s. then, as it follows from Lemma 4.5, the measures P and Q_θ are equivalent on \mathcal{F}_∞ . Since under \tilde{Q}_θ the spine process satisfies

$$\frac{\xi_t}{\int_0^t \lambda(\theta^+(s) - \theta^-(s)) ds} \rightarrow 1 \text{ a.s. as } t \rightarrow \infty$$

it would then follow that under P there is a particle with such asymptotic behaviour too. That would give the lower bound on the rightmost particle:

$$\liminf_{t \rightarrow \infty} \frac{R_t}{\int_0^t \lambda(\theta^+(s) - \theta^-(s)) ds} \geq 1,$$

which we can then optimise over suitable θ^+ and θ^- .

The upper bound on the rightmost particle needs a slightly different approach, which we present in the last subsection.

Remark 6.2. Let us note that the only important feature of $\theta^+(\cdot)$ in cases **B** and **C** is its asymptotic growth. By this we mean that we have freedom in defining $\theta(\cdot)$ as long as we keep

$$\theta^+(t) \sim \frac{c}{\lambda(1-p)} \frac{t^{b-1}}{(\log t)^b} \text{ as } t \rightarrow \infty \text{ in Case A}$$

and

$$\log \theta^+(t) \sim \alpha \sqrt{t} \text{ as } t \rightarrow \infty \text{ in Case B.}$$

Remark 6.3. Parts A ii), B ii) and C ii) of Theorem 6.1 will not be used in the proof of our main result, Theorem 1.2. We included them to better illustrate the behaviour of martingales M_θ .

Recall Lemma 4.5, which says that for events $A \in \mathcal{F}_\infty$

$$Q_\theta(A) = \int_A \limsup_{t \rightarrow \infty} M_\theta(t) dP + Q_\theta(A \cap \{\limsup_{t \rightarrow \infty} M_\theta(t) = \infty\}) \quad (6.1)$$

Immediate consequences of this (after taking $A = \Omega$) are:

1) $Q_\theta(\limsup_{t \rightarrow \infty} M_\theta(t) = \infty) = 1 \Leftrightarrow \limsup_{t \rightarrow \infty} M_\theta(t) = 0$ P -a.s. So to prove parts A ii), B ii) and C ii) of Theorem 6.1 we need to show that $\limsup_{t \rightarrow \infty} M_\theta(t) = \infty$ Q_θ -a.s.

2) $Q_\theta(\limsup_{t \rightarrow \infty} M_\theta(t) < \infty) = 1 \Leftrightarrow EM_\theta(\infty) = 1$ in which case $P(M_\theta(\infty) > 0) > 0$ and M_θ is L^1 -convergent w.r.t P as it follows from Scheffe's Lemma. Thus M_θ is P -uniformly integrable. So to prove the uniform integrability in parts A i), B i) and C i) of Theorem 6.1 we need to show that $\limsup_{t \rightarrow \infty} M_\theta(t) < \infty$ Q_θ -a.s.

The fact that $P(M_\theta(\infty) > 0) = 1$ (in parts A i), B i) and C i)) requires additionally a certain zero-one law, which we shall give at the end of this subsection.

Proof of Theorem 6.1: uniform integrability in A i), B i), C i). We start with proving that for the given values of θ in A i), B i) and C i) M_θ is UI. As we just said above, it is sufficient to prove that

$$\limsup_{t \rightarrow \infty} M_\theta(t) < \infty \text{ } Q_\theta\text{-a.s.} \quad (6.2)$$

for the given paths θ . We have already seen how to do this using the spine decomposition in Section 5. Just as before it is sufficient for us to check that

$$\limsup_{t \rightarrow \infty} E^{\tilde{Q}_\theta}(M_\theta(t) | \tilde{\mathcal{G}}_\infty) = \limsup_{t \rightarrow \infty} (\text{spine}(t) + \text{sum}(t)) < \infty \text{ } \tilde{Q}_\theta\text{-a.s.} \quad (6.3)$$

Let us outline the main steps of proving (6.3) in cases A, B and C.

Case A ($p = 0$), homogeneous branching:

We note that under \tilde{Q}_θ , $(\xi_t^+)_{t \geq 0} \stackrel{d}{=} PP(\lambda\theta_0)$ and $(\xi_t^-)_{t \geq 0} \stackrel{d}{=} PP(\frac{\lambda}{\theta_0})$. Hence

$$\frac{\xi_t^+}{t} \rightarrow \lambda\theta_0 \text{ and } \frac{\xi_t^-}{t} \rightarrow \frac{\lambda}{\theta_0} \text{ } \tilde{Q}_\theta\text{-a.s.}$$

Then using the above convergence results we wish to show that there exist some positive constant C'' and a \tilde{Q}_θ -a.s. finite time T' such that $\forall t > T'$

$$\text{spine}(t) \leq e^{-C''t}. \quad (6.4)$$

We observe that for any $\epsilon > 0$ there exists a \tilde{Q}_θ -a.s. finite time T_ϵ such that $\forall t > T_\epsilon$ $(1-\epsilon)\lambda\theta_0 t \leq \xi_t^+ \leq (1+\epsilon)\lambda\theta_0 t$ and $(1-\epsilon)\frac{\lambda}{\theta_0}t \leq \xi_t^- \leq (1+\epsilon)\frac{\lambda}{\theta_0}t$. Thus for $t > T_\epsilon$

$$\begin{aligned} spine(t) &\leq \exp\left(\lambda(1+\epsilon)\theta_0 \log \theta_0 t + \lambda(1-\epsilon)\frac{1}{\theta_0} \log\left(\frac{1}{\theta_0}\right)t + \lambda\left(2 - \theta_0 - \frac{1}{\theta_0}\right)t - \beta t\right) \\ &= \exp\left(\left(\lambda\left[g(\theta_0) + \epsilon\left(\theta_0 + \frac{1}{\theta_0}\right)\log \theta_0\right] - \beta\right)t\right), \end{aligned}$$

where

$$g(\theta) = \left(\theta - \frac{1}{\theta}\right) \log \theta - \left(\theta + \frac{1}{\theta}\right) + 2, \theta \in [1, \infty) \quad (6.5)$$

is an increasing function such that $g(\hat{\theta}) = \frac{\beta}{\lambda}$ (see the definition of $\hat{\theta}$). Then since $\theta_0 < \hat{\theta}$ it follows that for ϵ small enough

$$\lambda\left(g(\theta_0) + \epsilon\left(\theta_0 + \frac{1}{\theta_0}\right)\log \theta_0\right) - \beta < 0.$$

We thus take $T' = T_\epsilon$ for such an ϵ and $C'' = -\lambda\left(g(\theta_0) + \epsilon\left(\theta_0 + \frac{1}{\theta_0}\right)\log \theta_0\right) - \beta$ to obtain (6.4).

Then we have

$$\begin{aligned} sum(t) &= \sum_{u < node_t(\xi)} spine(S_u) \\ &\leq \left(\sum_{u < node_t(\xi), S_u \leq T'} spine(S_u)\right) + \left(\sum_{u < node_t(\xi), S_u > T'} e^{-C'' S_u}\right), \end{aligned}$$

where the first sum, call it C_1 , is \tilde{Q}_θ -a.s. bounded since it only counts births up to time T' . Thus

$$sum(t) \leq C_1 + \sum_{n=1}^{\infty} e^{-C'' S_n}, \quad (6.6)$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t \in [0, \infty)}$ is a Poisson process with rate 2β . Therefore $t^{-1}n_t \rightarrow 2\beta$ \tilde{Q}_θ -a.s. as $t \rightarrow \infty$ and hence $n^{-1}S_n \rightarrow (2\beta)^{-1}$ \tilde{Q}_θ -a.s. as $n \rightarrow \infty$. So for some \tilde{Q}_θ -a.s. finite positive random variable C_2 we have $S_n \geq C_2 n \quad \forall n$. Then substituting this into (6.6) we get

$$sum(t) \leq C_1 + \sum_{n=1}^{\infty} e^{-C'' C_2 n} < \infty \quad \tilde{Q}_\theta\text{-a.s.},$$

which gives (6.3).

Case B ($p \in (0, 1)$), inhomogeneous subcritical branching:

From Proposition 3.6 under \tilde{Q}_θ the process $(\xi_t^+)_{t \in [0, \infty)}$ is a time-inhomogeneous Poisson process with jump rate $\lambda\theta^+(t)$ and $(\xi_t^-)_{t \in [0, \infty)}$ is a Poisson process of rate λ . Then from Propositions 3.4 and 3.5 we find that, \tilde{Q}_θ -a.s.,

$$\frac{\xi_t^+}{\int_0^t \lambda\theta^+(s)ds} \rightarrow 1, \quad \frac{\xi_t^-}{\lambda t} \rightarrow 1, \quad \frac{\int_0^t \log \theta^+(s) d\xi_s^+}{\int_0^t \log \theta^+(s) \lambda\theta^+(s)ds} \rightarrow 1.$$

It can then be checked in a similar way as before that there exist some \tilde{Q}_θ -a.s. finite positive random variables C' , C''' and T' such that, $\forall t > T'$,

$$spine(t) \leq C' \exp\left(-C''' \int_0^t \frac{s^{\hat{b}p}}{(\log(s+2))^{\hat{b}p}} ds\right).$$

For the sum term of the spine decomposition we have when $t > T'$

$$sum(t) \leq \sum_{\substack{u < node_t(\xi), \\ S_u \leq T'}} spine(S_u) + \sum_{\substack{u < node_t(\xi), \\ S_u > T'}} C' \exp \left(-C'' \int_0^{S_u} \frac{s^{\hat{b}p}}{(\log(s+2))^{\hat{b}p}} ds \right)$$

The first sum is a \tilde{Q}_θ -a.s. finite random variable which doesn't depend on t , and which we call C_1 . Then

$$sum(t) \leq C_1 + C' \sum_{n=1}^{\infty} \exp \left(-C'' \int_0^{S_n} \frac{s^{\hat{b}p}}{(\log(s+2))^{\hat{b}p}} ds \right), \quad (6.7)$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t \in [0, \infty)}$ conditional on the path of the spine is time-inhomogeneous Poisson process (or Cox process) with jump rate $2\beta|\xi_t|^p$ at time t . Thus, we find

$$n_t \sim 2\beta \int_0^t |\xi_s|^p ds \sim 2\beta \left(\frac{c}{\hat{b}(1-p)} \right)^p \int_0^t \frac{s^{\hat{b}p}}{(\log(s+2))^{\hat{b}p}} ds \quad \tilde{Q}_\theta\text{-a.s. as } t \rightarrow \infty.$$

So for some \tilde{Q}_θ -a.s. finite positive random variable C_2 we have

$$\int_0^{S_n} \frac{s^{\hat{b}p}}{(\log(s+2))^{\hat{b}p}} ds \geq C_2 n \quad \forall n.$$

Then substituting this into (6.7) we verify that (6.3) again holds.

Case C ($p = 1$), inhomogeneous near-critical branching:

As in the previous case, under \tilde{Q}_θ the process $(\xi_t^+)_{t \in [0, \infty)}$ is a time-inhomogeneous Poisson process with jump rate $\lambda\theta^+(t)$ and $(\xi_t^-)_{t \in [0, \infty)}$ is a Poisson process of rate λ . Then \tilde{Q}_θ -a.s. we have

$$\frac{\xi_t^+}{\int_0^t \lambda\theta^+(s) ds} \rightarrow 1, \quad \frac{\xi_t^-}{\lambda t} \rightarrow 1, \quad \frac{\int_0^t \log \theta^+(s) d\xi_s^+}{\int_0^t \log \theta^+(s) \lambda\theta^+(s) ds} \rightarrow 1.$$

One can check that there exist some \tilde{Q}_θ -a.s. finite positive random variables C' , C'' and T' such that, $\forall t > T'$,

$$spine(t) \leq C' \exp \left(-C'' \int_0^t \sqrt{s} e^{\alpha\sqrt{s}} ds \right).$$

Then for $t > T'$

$$sum(t) \leq C_1 + C' \sum_{n=1}^{\infty} \exp \left(-C'' \int_0^{S_n} \sqrt{s} e^{\alpha\sqrt{s}} ds \right), \quad (6.8)$$

where $C_1 < \infty$ and S_n is the time of the n^{th} birth on the spine. The birth process along the spine $(n_t)_{t \in [0, \infty)}$ then satisfies

$$n_t \sim \int_0^t 2\beta|\xi_s| ds \sim \frac{4\beta\lambda}{\alpha} \int_0^t \sqrt{s} e^{\alpha\sqrt{s}} ds \quad \tilde{Q}_\theta\text{-a.s. as } t \rightarrow \infty.$$

So for some \tilde{Q}_θ -a.s. finite positive random variable C_2 we have

$$\int_0^{S_n} \sqrt{s} e^{\alpha\sqrt{s}} ds \geq C_2 n \quad \forall n.$$

Then substituting this into (6.8) we again find that (6.3) holds.

Thus we have completed the proof of uniform integrability and the fact that $P(M_\theta(\infty) > 0) > 0$ in Theorem 6.1. \square

Proof of Theorem 6.1: parts A ii), B ii), C ii). Since one of the particles at time t is the spine, we have

$$\begin{aligned} M_\theta(t) \geq & \exp \left(\int_0^t \log(\theta^+(s)) d\xi_s^+ + \int_0^t \log(\theta^-(s)) d\xi_s^- \right. \\ & \left. + \lambda \int_0^t (2 - \theta^+(s) - \theta^-(s)) ds - \beta \int_0^t |\xi_s|^p ds \right) = \text{spine}(t). \end{aligned}$$

For the paths θ in parts ii) of Theorem 6.1 one can check (following the same analysis as in the proof of parts i) of the Theorem) that $\text{spine}(t) \rightarrow \infty$ \tilde{Q}_θ -a.s. Thus

$$\limsup_{t \rightarrow \infty} M_\theta(t) = \infty \quad \tilde{Q}_\theta\text{-a.s.}$$

and so also Q_θ -a.s. Recalling (6.1) we see that $M_\theta(\infty) = 0$ P -a.s. for the proposed choices of θ . \square

It remains to show that in Theorem 6.1 $P(M_\theta(\infty) > 0) = 1$ when M_θ is UI. The following 0-1 law will do the job.

Lemma 6.4. *Let $q : \mathbb{Z} \rightarrow [0, 1]$ be such that $M_t := \prod_{u \in N_t} q(X_u(t))$ is a P -martingale (usually referred to as a product martingale). Then $q(x) \equiv q \in \{0, 1\}$.*

Proof of Lemma 6.4. Since M_t is a martingale and one of the particles alive at time t is the spine we have

$$q(x) = E^x M_t = \tilde{E}^x M_t \leq \tilde{E}^x q(\xi_t).$$

So $q(\xi_t)$ is a positive \tilde{P} -submartingale. Since it is bounded it converges \tilde{P} -a.s. to some limit q_∞ . We also know that under \tilde{P} , $(\xi_t)_{t \geq 0}$ is a continuous-time random walk. Recurrence of $(\xi_t)_{t \geq 0}$ implies that $q_\infty \equiv q(0)$ and that $q(x)$ is constant in x .

Now suppose for contradiction that $q(0) \in (0, 1)$. Then

$$M_t = \prod_{u \in N_t} q(X_u(t)) = q(0)^{|N_t|} \rightarrow 0$$

because $|N_t| \rightarrow \infty$. Since M is bounded it is uniformly integrable, so $q(0) = EM_\infty = 0$, which is a contradiction. So $q(0) \notin (0, 1)$ and thus $q(0) \in \{0, 1\}$. \square

Proof of Theorem 6.1: positivity of limits in A i), B i), C i). We apply Lemma 6.4 to $q(x) = P^x(M_\theta(\infty) = 0)$. By the tower property of conditional expectations and the branching Markov property we have

$$q(x) = E^x \left(P^x(M_\theta(\infty) = 0 | \mathcal{F}_t) \right) = E^x \left(\prod_{u \in N_t} q(X_u(t)) \right)$$

whence $\prod_{u \in N_t} q(X_u(t))$ is a P -martingale. Also $E(M_\theta(\infty)) = M_\theta(0) = 1 > 0$. Therefore $P(M_\theta(\infty) = 0) \neq 1$. So by Lemma 6.4 $P(M_\theta(\infty) = 0) = 0$. \square

6.2 Lower bound on the rightmost particle

Proposition 6.5. *Let $\hat{\theta}$, \hat{b} and \hat{c} be as defined in Theorem 1.2. Then*

Case A ($p = 0$):

$$\liminf_{t \rightarrow \infty} \frac{R_t}{t} \geq \lambda(\hat{\theta} - \frac{1}{\hat{\theta}}) \quad P\text{-a.s.}$$

Case B ($p \in (0, 1)$):

$$\liminf_{t \rightarrow \infty} \left(\frac{\log t}{t} \right)^{\hat{b}} R_t \geq \hat{c} \text{ } P\text{-a.s.}$$

Case C ($p = 1$):

$$\liminf_{t \rightarrow \infty} \frac{\log R_t}{\sqrt{t}} \geq \sqrt{2\beta} \text{ } P\text{-a.s.}$$

Proof.

Case A ($p = 0$):

We consider $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_0$, $\theta^-(\cdot) \equiv \frac{1}{\theta_0}$ and $\theta_0 < \hat{\theta}$. Take the event

$$B_{\theta_0} := \left\{ \exists \text{ infinite line of descent } u : \liminf_{t \rightarrow \infty} \frac{X_u(t)}{t} = \lambda(\theta_0 - \frac{1}{\theta_0}) \right\} \in \mathcal{F}_{\infty}.$$

We know that $\tilde{Q}_{\theta}(\lim_{t \rightarrow \infty} \frac{\xi_t}{t} = \lambda(\theta_0 - \frac{1}{\theta_0})) = 1$. Hence $Q_{\theta}(B_{\theta_0}) = \tilde{Q}_{\theta}(B_{\theta_0}) = 1$. Since Q_{θ} and P are equivalent it follows that $P(B_{\theta_0}) = 1$. Thus $P(\liminf_{t \rightarrow \infty} t^{-1} R_t \geq \lambda(\theta_0 - \theta_0^{-1})) = 1$. Taking the limit $\theta_0 \nearrow \hat{\theta}$ we get

$$P\left(\liminf_{t \rightarrow \infty} \frac{R_t}{t} \geq \lambda(\hat{\theta} - \frac{1}{\hat{\theta}})\right) = 1.$$

Case B ($p \in (0, 1)$):

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, $\theta^+(s) = \frac{c}{\lambda(1-p)} \frac{s^{\hat{b}-1}}{(\log(s+2))^{\hat{b}}}$ and $c < \hat{c}$. Take the event

$$B_c := \left\{ \exists \text{ infinite line of descent } u : \liminf_{t \rightarrow \infty} \left(\frac{\log t}{t} \right)^{\hat{b}} X_u(t) = c \right\}.$$

Same argument as above gives that $P(B_c) = 1$ and hence $P\left(\liminf_{t \rightarrow \infty} \left(t^{-1} \log t \right)^{\hat{b}} R_t \geq c\right) = 1$ for all $c < \hat{c}$. Letting $c \nearrow \hat{c}$ proves the result.

Case C ($p = 1$):

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, $\theta^+(s) = e^{\alpha\sqrt{s}}$ and $\alpha < \sqrt{2\beta}$. Take the event

$$B_{\alpha} := \left\{ \exists \text{ infinite line of descent } u : \liminf_{t \rightarrow \infty} \frac{\log X_u(t)}{\sqrt{t}} = \sqrt{2\beta} \right\}.$$

Again, the same argument as above gives $P(B_{\alpha}) = 1$ and hence for all $\alpha < \sqrt{2\beta}$ we find that $P\left(\liminf_{t \rightarrow \infty} t^{-1/2} \log R_t \geq \alpha\right) = 1$. Letting $\alpha \nearrow \sqrt{2\beta}$ proves the result. \square

6.3 Upper bound on the rightmost particle

To complete the proof of Theorem 1.2 and hence the whole section we need to prove the following proposition.

Proposition 6.6. *Let $\hat{\theta}$, \hat{b} and \hat{c} be as defined in Theorem 1.2. Then for different values of p we have the following.*

Case A ($p = 0$):

$$\limsup_{t \rightarrow \infty} \frac{R_t}{t} \leq \lambda(\hat{\theta} - \frac{1}{\hat{\theta}}) \text{ } P\text{-a.s.}$$

Case B ($p \in (0, 1)$):

$$\limsup_{t \rightarrow \infty} \left(\frac{\log t}{t} \right)^{\hat{b}} R_t \leq \hat{c} \text{ } P\text{-a.s.}$$

Case C ($p = 1$):

$$\limsup_{t \rightarrow \infty} \frac{\log R_t}{\sqrt{t}} \leq \sqrt{2\beta} \text{ } P\text{-a.s.}$$

To prove Proposition 6.6 we shall assume for contradiction that it is false. Then we shall show that under such assumption certain additive P -martingales will diverge to ∞ contradicting the Martingale Convergence Theorem.

We start by proving the following 0-1 law.

Lemma 6.7. *Let $g : [0, \infty) \rightarrow \mathbb{R}$ be increasing, $f : [0, \infty) \rightarrow [0, \infty)$ be such that $\forall s \geq 0 \frac{f(t)}{f(s+t)} \rightarrow 1$ as $t \rightarrow \infty$ and $a > 0$. Then*

$$P\left(\limsup_{t \rightarrow \infty} \frac{g(R_t)}{f(t)} \leq a\right) \in \{0, 1\}.$$

Proof. We consider

$$q(x) = P^x\left(\limsup_{t \rightarrow \infty} \frac{g(R_t)}{f(t)} \leq a\right).$$

Then, it is easy to see that

$$\begin{aligned} q(x) &= E^x\left(P^x\left(\limsup_{t \rightarrow \infty} \frac{g(R_{t+s})}{f(t+s)} \leq a \mid \mathcal{F}_s\right)\right) \\ &= E^x\left(P^x\left(\limsup_{t \rightarrow \infty} \frac{g(\max_{u \in N_s} R_t^u)}{f(t+s)} \leq a \mid \mathcal{F}_s\right)\right) \\ &= E^x\left(P^x\left(\max_{u \in N_s} \left\{\limsup_{t \rightarrow \infty} \frac{g(R_t^u)}{f(t+s)}\right\} \leq a \mid \mathcal{F}_s\right)\right) \\ &= E^x\left(\prod_{u \in N_s} P^{X_u(s)}\left(\limsup_{t \rightarrow \infty} \frac{g(R_t)}{f(t+s)} \leq a\right)\right) \\ &= E^x\left(\prod_{u \in N_s} P^{X_u(s)}\left(\limsup_{t \rightarrow \infty} \frac{g(R_t)}{f(t)} \leq a\right)\right) \\ &= E^x\left(\prod_{u \in N_s} q(X_u(s))\right), \end{aligned}$$

where $(R_t^u)_{t \geq 0}$ is the position of the rightmost particle of a subtree started from $X_u(s)$.

Thus $\prod_{u \in N_t} q(X_u(t))$ is a martingale. Applying Lemma 6.4 to $q(\cdot)$ we obtain the required result. \square

Proof of Proposition 6.6. The first step of the proof is slightly different for cases A, B and C, so we do it for the three cases separately.

Case A ($p = 0$)

Let us suppose for contradiction that $\exists \theta_0 > \hat{\theta}$ such that

$$P\left(\limsup_{t \rightarrow \infty} \frac{R_t}{t} > \lambda(\theta_0 - \frac{1}{\theta_0})\right) = 1. \quad (6.9)$$

Choose any $\theta_A \in (\hat{\theta}, \theta_0)$ and take $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_A$, $\theta^-(\cdot) = \frac{1}{\theta_A}$. Let

$$f_A(s) := \lambda(\theta_A - \frac{1}{\theta_A})s, \quad s \geq 0.$$

Case B ($p \in (0, 1)$)

Let us suppose for contradiction that $\exists c_0 > \hat{c}$ such that

$$P\left(\limsup_{t \rightarrow \infty} \left(\frac{\log t}{t}\right)^{\hat{b}} R_t > c_0\right) = 1. \quad (6.10)$$

Choose any $c_1 \in (\hat{c}, c_0)$ and take $\theta = (\theta^+, \theta^-)$, where $\theta^+(s) = \theta_B(s)$, $\theta^-(s) = \frac{1}{\theta_B(s)}$ and

$$\theta_B(s) = \frac{c_1}{\lambda(1-p)} \frac{s^{\hat{b}-1}}{(\log(s+2))^{\hat{b}}}, \quad s \geq 0.$$

Let

$$f_B(s) := c_1 \left(\frac{s}{\log(s+2)}\right)^{\hat{b}}, \quad s \geq 0.$$

Case C ($p = 1$)

Let us suppose for contradiction that $\exists \alpha_0 > \sqrt{2\beta}$ such that

$$P\left(\limsup_{t \rightarrow \infty} \frac{\log R_t}{\sqrt{t}} > \alpha_0\right) = 1. \quad (6.11)$$

Choose any $\alpha_1 \in (\sqrt{2\beta}, \alpha_0)$ and take $\theta = (\theta^+, \theta^-)$, where $\theta^+(s) = \theta_C(s)$, $\theta^-(s) = \frac{1}{\theta_C(s)}$ and

$$\theta_C(s) = \frac{1}{\sqrt{s+1}} e^{\alpha_1 \sqrt{s}}, \quad s \geq 0.$$

Let

$$f_C(s) := e^{\alpha_1 \sqrt{s}}, \quad s \geq 0.$$

The next step in the proof is the same in all cases.

Let us write f to denote f_A , f_B and f_C . We define $D(f)$ to be the space-time region bounded above by the curve $y = f(t)$ and below by the curve $y = -f(t)$.

Under P the spine process $(\xi_t)_{t \geq 0}$ is a continuous-time random walk and so $\frac{|\xi_t|}{t} \rightarrow 0$ P -a.s. as $t \rightarrow \infty$. Hence there exists an a.s. finite random time $T' < \infty$ such that $\xi_t \in D(f)$ for all $t > T'$.

Since $(\xi_t)_{t \geq 0}$ is recurrent it will spend an infinite amount of time at position $y = 1$. During this time it will be giving birth to offspring at rate β . This assures us of the existence of an infinite sequence $\{T_n\}_{n \in \mathbb{N}}$ of birth times along the path of the spine when it stays at $y = 1$ with $0 \leq T' \leq T_1 < T_2 < \dots$ and $T_n \nearrow \infty$.

Denote by u_n the label of the particle born at time T_n , which does not continue the spine. Then each particle u_n gives rise to an independent copy of the Branching random walk under P started from ξ_{T_n} at time T_n . Almost surely, by assumptions (6.9), (6.10) and (6.11), each u_n has some descendant that leaves the space-time region $D(f)$.

Let $\{v_n\}_{n \in \mathbb{N}}$ be the subsequence of $\{u_n\}_{n \in \mathbb{N}}$ of those particles whose first descendent leaving $D(f)$ does this by crossing the upper boundary $y = f(t)$. Since the breeding potential is symmetric and the particles u_n are born in the upper half-plane, there is at least probability $\frac{1}{2}$ that the first descendant of u_n to leave $D(f)$ does this by crossing the positive boundary curve. Therefore P -a.s. the sequence $\{v_n\}_{n \in \mathbb{N}}$ is infinite.

Let w_n be the descendent of v_n , which exits $D(f)$ first and let J_n be the time when this occurs. That is,

$$J_n = \inf \{t : X_{w_n}(t) \geq f(t)\}.$$

Note that the path of particle w_n satisfies

$$|X_{w_n}(s)| < f(s) \quad \forall s \in [T', J_n].$$

Clearly $J_n \rightarrow \infty$ as $n \rightarrow \infty$. To obtain a contradiction we shall show that the additive martingale M_θ fails to converge along the sequence of times $\{J_n\}_{n \geq 1}$, where θ was defined above differently for cases A, B and C. Thus for the last bit of the proof we have to look at cases A, B and C separately again.

Case A ($p = 0$)

$$\begin{aligned} M_\theta(J_n) &= \sum_{u \in N_{J_n}} \exp \left\{ \int_0^{J_n} \log \theta_A dX_u^+(s) + \int_0^{J_n} \log \left(\frac{1}{\theta_A} \right) dX_u^-(s) \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_A - \frac{1}{\theta_A} \right) ds - \beta \int_0^{J_n} 1 ds \right\} \\ &\geq \exp \left\{ \int_0^{J_n} \log \theta_A dX_{w_n}^+(s) + \int_0^{J_n} \log \left(\frac{1}{\theta_A} \right) dX_{w_n}^-(s) \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_A - \frac{1}{\theta_A} \right) ds - \beta \int_0^{J_n} 1 ds \right\} \\ &= \exp \left\{ \log \theta_A X_{w_n}^+(J_n) - \log \theta_A X_{w_n}^-(J_n) + \lambda \left(2 - \theta_A - \frac{1}{\theta_A} \right) J_n - \beta J_n \right\} \\ &= \exp \left\{ \log \theta_A X_{w_n}(J_n) + \lambda \left(2 - \theta_A - \frac{1}{\theta_A} \right) J_n - \beta J_n \right\} \\ &\geq \exp \left\{ a_1 J_n \log \theta_A + \lambda \left(2 - \theta_A - \frac{1}{\theta_A} \right) J_n - \beta J_n \right\} \\ &= \exp \left\{ \left(\lambda \left(\theta_A - \frac{1}{\theta_A} \right) \log \theta_A + \lambda \left(2 - \theta_A - \frac{1}{\theta_A} \right) - \beta \right) J_n \right\} \\ &= \exp \left\{ \left(\lambda g(\theta_A) - \beta \right) J_n \right\}, \end{aligned}$$

where $g(\cdot)$ is the same as in (6.5). Then since $g(\cdot)$ is increasing, $\theta_A > \hat{\theta}$ and $g(\hat{\theta}) = \frac{\beta}{\lambda}$ it follows that

$$\lambda g(\theta_A) - \beta > 0$$

and thus $M_\theta(J_n) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Therefore assumption (6.9) is wrong and we must have that $\forall \theta_0 > \hat{\theta}$

$$P \left(\limsup_{t \rightarrow \infty} \frac{R_t}{t} > \lambda \left(\theta_0 - \frac{1}{\theta_0} \right) \right) \neq 1.$$

It follows from Lemma 6.7 that $\forall \theta_0 > \hat{\theta} P\left(\limsup_{t \rightarrow \infty} \frac{R_t}{t} > \lambda(\theta_0 - \frac{1}{\theta_0})\right) = 0$. Hence $P\left(\limsup_{t \rightarrow \infty} \frac{R_t}{t} \leq \lambda(\theta_0 - \frac{1}{\theta_0})\right) = 1$ and after letting $\theta_0 \searrow \hat{\theta}$ we get

$$P\left(\limsup_{t \rightarrow \infty} \frac{R_t}{t} \leq \lambda(\hat{\theta} - \frac{1}{\hat{\theta}})\right) = 1.$$

Case B ($p \in (0, 1)$)

$$\begin{aligned} M_{\theta}(J_n) &= \sum_{u \in N_{J_n}} \exp \left\{ \int_0^{J_n} \log \theta_B(s) dX_u^+(s) + \int_0^{J_n} \log \left(\frac{1}{\theta_B(s)} \right) dX_u^-(s) \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_B(s) - \frac{1}{\theta_B(s)} \right) ds - \beta \int_0^{J_n} |X_u(s)|^p ds \right\} \\ &\geq \exp \left\{ \int_0^{J_n} \log \theta_B(s) dX_{w_n}^+(s) + \int_0^{J_n} \log \left(\frac{1}{\theta_B(s)} \right) dX_{w_n}^-(s) \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_B(s) - \frac{1}{\theta_B(s)} \right) ds - \beta \int_0^{J_n} |X_{w_n}(s)|^p ds \right\}. \end{aligned}$$

Applying the integration by parts formula from Proposition 3.3 we get

$$\begin{aligned} &\exp \left\{ \log \theta_B(J_n) X_{w_n}^+(J_n) - \int_0^{J_n} \frac{\theta'_B(s)}{\theta_B(s)} X_{w_n}^+(s) ds \right. \\ &\quad \left. - \log \theta_B(J_n) X_{w_n}^-(J_n) + \int_0^{J_n} \frac{\theta'_B(s)}{\theta_B(s)} X_{w_n}^-(s) ds \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_B(s) - \frac{1}{\theta_B(s)} \right) ds - \beta \int_0^{J_n} |X_{w_n}(s)|^p ds \right\} \\ &= \exp \left\{ \log \theta_B(J_n) X_{w_n}(J_n) - \int_0^{J_n} \frac{\theta'_B(s)}{\theta_B(s)} X_{w_n}(s) ds \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_B(s) - \frac{1}{\theta_B(s)} \right) ds - \beta \int_0^{J_n} |X_{w_n}(s)|^p ds \right\} \\ &\geq C \exp \left\{ \log \theta_B(J_n) f_B(J_n) - \int_0^{J_n} \frac{\theta'_B(s)}{\theta_B(s)} f_B(s) ds \right. \\ &\quad \left. + \lambda \int_0^{J_n} \left(2 - \theta_B(s) - \frac{1}{\theta_B(s)} \right) ds - \beta \int_0^{J_n} f_B(s)^p ds \right\} \end{aligned}$$

using the facts that $X_{w_n}(J_n) \geq f_B(J_n)$ and $|X_{w_n}(s)| < f_B(s)$ for $s \in [T', J_n)$ and where C is some P -a.s positive random variable. Now asymptotic properties of $\theta_B(\cdot)$ and $f_B(\cdot)$ give us that for any $\epsilon > 0$ and n large enough the above expression is

$$\geq C_{\epsilon} \exp \left\{ (\hat{b} - 1) c_1 \frac{(J_n)^{\hat{b}}}{(\log J_n)^{\hat{b}-1}} (1 - \epsilon) - \beta c_1^p \frac{1}{\hat{b}} \frac{(J_n)^{\hat{b}}}{(\log J_n)^{\hat{b}-1}} (1 + \epsilon) \right\}$$

for some P -a.s. positive random variable C_{ϵ} . Then since $c_1 > \hat{c} = \left(\frac{\beta(1-p)^2}{p} \right)^{(1-p)^{-1}}$

$$(\hat{b} - 1) c_1 (1 - \epsilon) - \beta c_1^p \frac{1}{\hat{b}} (1 + \epsilon) = c_1^p (\hat{b} - 1) (1 - \epsilon) \left(c_1^{1-p} - \hat{c}^{1-p} \frac{1 + \epsilon}{1 - \epsilon} \right) > 0$$

for ϵ small enough. Thus $M_\theta(J_n) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction. Therefore assumption (6.10) is wrong and we must have that $\forall c_0 > \hat{c}$

$$P\left(\limsup_{t \rightarrow \infty} \left(\frac{\log t}{t}\right)^{\hat{b}} R_t > c_0\right) \neq 1.$$

It follows from Lemma 6.7 that $\forall c_0 > \hat{c}$

$$P\left(\limsup_{t \rightarrow \infty} \left(\frac{\log t}{t}\right)^{\hat{b}} R_t \leq c_0\right) = 1$$

Hence taking the limit $c_0 \searrow \hat{c}$ proves Proposition 6.6 in Case B.

Case C ($p = 1$)

Essentially the same argument as in Case B gives that for any $\epsilon > 0$ and n large enough

$$M_\theta(J_n) \geq C_\epsilon \exp \left\{ (1 - \epsilon)\alpha_1 \sqrt{J_n} e^{\alpha_1 \sqrt{J_n}} - (1 + \epsilon) \frac{2\beta}{\alpha_1} \sqrt{J_n} e^{\alpha_1 \sqrt{J_n}} \right\}$$

for some $C_\epsilon > 0$ P -a.s. Then since $\alpha_1 > \sqrt{2\beta}$

$$(1 - \epsilon)\alpha_1 - (1 + \epsilon) \frac{2\beta}{\alpha_1} > 0$$

for ϵ chosen sufficiently small. Therefore $M_\theta(J_n) \rightarrow \infty$, which is a contradiction. Hence $\forall \alpha_0 > \sqrt{2\beta}$

$$P\left(\limsup_{t \rightarrow \infty} \frac{\log R_t}{\sqrt{t}} \leq \alpha_0\right) = 1$$

and therefore

$$P\left(\limsup_{t \rightarrow \infty} \frac{\log R_t}{\sqrt{t}} \leq \sqrt{2\beta}\right) = 1.$$

This finishes the proof of Proposition 6.6 and also Theorem 1.2 □

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